Abstract

An analysis of steady, two dimensional, Laminar, boundary-layer flow of a viscous, incompressible, electrically conducting and radiating fluid towards an exponentially stretching porous sheet embedded in a doubly stratified medium in the presence of a uniform transverse magnetic field with heat and mass transfer is presented. The Rossland approximation is used to describe the radiative heat transfer in the limit of optically thick fluids. The effect of internal heat generation/absorption is also taken into account. Similarity transformations were used to convert the partial differential equations corresponding to the momentum, energy and concentration into highly nonlinear ordinary differential equations. Numerical solutions of these equations are obtained by fourth order Runge-Kutta method along with shooting technique. It is found that the temperature and concentration decreases with the increase of thermal and salutal stratification parameters respectively. The heat transfer rate increases with Prandtl number, but decreases with radiation parameter.
1. Introduction

The study of mixed convective transport in a doubly stratified (thermal and/or salutal stratification) fluid saturated porous medium has many industrial and engineering applications such as heat rejection into the environment such as lakes, rivers, and seas, thermal energy storage systems such as solar ponds, and heat transfer from thermal sources such as the condensers of power plants. Stratification of fluid means deposition or formation of layers and occurs due to temperature variations, concentration differences, or the presence of different fluids. It is necessary to investigate the temperature stratification and concentration differences of hydrogen and oxygen in lakes and ponds as they may directly affect the growth rate of all cultured species. Mukhopadhyay and Ishak [1] analyzed the mixed convection flow of a viscous and incompressible fluid towards a stretching cylinder immersed in a thermally stratified medium. Hassanien et al. [2] presented the influence of thermal dispersion and stratification on the flow and temperature fields for mixed convection from a vertical plate embedded in a porous medium.

The study of boundary layer flow on a continuous stretching sheet has attracted considerable interest during the last few decades due to its ever increasing industrial applications such as hot rolling, wire drawing, glass-fiber and paper production, drawing of plastic films, metal and polymer extrusion and metal spinning. Both the simultaneous heating or cooling during such processes and the kinematics of stretching has a decisive influence on the quality of the final products [3] (Magyari & Keller 1999).

The boundary layer flow caused by a stretching sheet which moves with a velocity varying linearly with the distance from a fixed point was first considered by Crane [4]. Partha et al. [5] investigated the effect of viscous dissipation on the mixed convection heat transfer from an exponentially stretching surface. Recently, Sajid and Hayat [6] investigated the radiation effects on the flow over an exponentially stretching sheet analytically by using the homotopy analysis method. The numerical solution for this problem was then given by Bidin and Nazar [7]. Recently, Swati Mukhopadhyaya [8] analyzed the MHD boundary layer flow and heat transfer over an exponentially stretching sheet embedded in a thermally stratified medium. Later, she extended her work by including mass transfer and chemical reaction effects [9]. D.Srinivasacharya et al. [10] studied the Non-Darcy mixed convection in a doubly stratified porous medium with soret-Dufour effects.

The study of heat generation in moving fluids is important in view of several physical problems such as those dealing with chemical reactions and those concerned with dissociating fluids. To the best of our knowledge, the problem of steady MHD convective heat and mass transfer flow of viscous, incompressible, electrically conducting and radiating fluid flow towards an exponentially stretching porous sheet embedded in a doubly stratified medium with internal heat generation / absorption and radiation has remained unexplored. So, the main objective of this paper is to extend the work of Swati Mukhopadhyaya [8] in four directions: (i) to consider mass transfer effects, (ii) to consider the radiation effects (iii) to consider the internal heat generation/ absorption effect and (iv) to include the salutal stratification effect. The governing equations of the flow are solved numerically, and the effects of various flow parameters on the flow field have been discussed. The organization of the paper is as follows. In Section 2, we describe the model with its governing equations and boundary conditions. In section 3, we present method of solution, In Section 4, we present results and discussion. A comparison is made with the available results in the literature, and salient features of the new results are analyzed and discussed. Finally, in Section 5, we summarize our results and present our conclusions.
2. Mathematical Analysis

A steady two dimensional laminar mixed convective boundary layer flow of a viscous incompressible, electrically conducting and radiating fluid past an exponentially stretching porous sheet coinciding with the plane \( y = 0 \) in a doubly stratified medium is considered. The flow is confined to \( y > 0 \). Two equal and opposite forces are applied along the x-axis, so that the wall is stretched keeping the origin fixed (see Fig.1). A variable magnetic field \( B(x) = B_0 e^{\frac{x}{x}} \) is applied normal to the sheet, \( B_0 \) being a constant. [11]. It is assumed that there is no applied voltage, which implies the absence of an electric field. The transverse applied magnetic field and magnetic Reynolds number are assumed to be very small so that the induced magnetic field and the Hall effects are negligible. The concentration of the diffusing species in the binary mixture is assumed to be very small in comparison with the other chemical species which are present, hence soret and dufour effects are negligible. The sheet is embedded in a stratified medium of variable ambient temperature in the form \( T_\infty(x) = T_0 + A_2 e^{\frac{2x}{x}} \) and concentration in the form \( C_\infty(x) = C_0 + B_2 e^{\frac{2x}{x}} \), where \( T_\infty(x) > T_\infty(x) \) and \( C_\infty(x) > C_\infty(x) \), \( T_\infty(x) \) and \( C_\infty(x) \) being the surface temperature and concentration respectively. It is assumed that \( T_\infty(x) = T_0 + A_1 e^{\frac{2x}{x}} \) and \( C_\infty(x) = C_0 + B_1 e^{\frac{2x}{x}} \), where \( T_0 \) and \( C_0 \) are the reference temperature and concentration respectively and \( A_1, B_1 > 0 \) and \( A_2, B_2 \geq 0 \) are constants. Further, we assume that there is a temperature dependent heat source in the boundary layer.

![Figure 1. Physical model and coordinate system of the problem](image)

The continuity, momentum, energy and concentration equations governing such type of flow are written as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

\[
u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} - \frac{\sigma B(x)^2 u}{\rho} \tag{2}
\]

\[
u \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p} \frac{\partial q_r}{\partial y} + Q(T - T_\infty) \tag{3}
\]

\[
u \frac{\partial C}{\partial x} + \frac{\partial C}{\partial y} = D_m \frac{\partial^2 C}{\partial y^2} - K_r(C - C_\infty) \tag{4}
\]
where \( u \) and \( v \) are the components of velocity in the \( x \) and \( y \) directions respectively, \( \nu \) is the kinematic viscosity, \( \rho \) is the fluid density, \( \sigma \) is the electrical conductivity of the fluid, \( k \) is the thermal conductivity, \( c_p \) is the specific heat at constant pressure, \( q_r \) is the radiative heat flux, \( D \) is the solute diffusion coefficient. The second and third terms on the right hand side of the energy equation (3) denote the radiative heat flux term and a temperature dependent heat source/sink term.

The boundary conditions for the velocity, temperature and concentration fields are

\[
\begin{align*}
 u & = U , \quad v = -V(x), \quad T = T_w(x), \quad C = C_w(x) \quad \text{at } y = 0 \\
 u \to 0, T = T_\infty(x), C = C_\infty(x) \quad \text{as } y \to \infty
\end{align*}
\]  

where \( U = U_0 e^{x^2/\lambda^2} \) is the stretching velocity, \( U_0 \) is the reference velocity, \( V(x) > 0 \) is the velocity of suction and \( V(x) < 0 \) is the velocity of blowing, \( V(x) = V_0 e^{x^2/\lambda^2} \) is a special type of velocity at the wall, where \( V_0 \) is the initial strength of suction, \( Q = \frac{1}{2} Q_0 e^{x^2/\lambda^2} \) is the variable volumetric rate of heat generation (i.e., heat source) or heat absorption (i.e., heat sink), where \( Q_0 \) is a constant having the same dimension as \( Q \), \( k_r = \frac{1}{2} k_0 e^{x^2/\lambda^2} \) is the variable rate of chemical conversion of the first order irreversible reaction, where \( k_0 \) is a constant having the same dimension as \( k \), the diffusivity of the species can either be destroyed or generated in the reaction.

### 3. Method of Solution

Introducing the suitable transformations as

\[
\eta = \sqrt{\frac{2}{\lambda^2} U_0^2} e^{\frac{x^2}{2\lambda^2}}, \quad u = U_0 f'(\eta) e^{\frac{x^2}{2\lambda^2}}, \quad v = -e^{\frac{x^2}{2\lambda^2}} \left( f + \eta f' \right)
\]

\[
R = \frac{4\sigma T_\infty^2}{kk\lambda^2}, \quad Pr = \frac{\rho c_p \mu}{k}, \quad Sc = \frac{\nu}{D'}, \quad \epsilon_1 = \frac{\alpha}{A_1}, \quad \epsilon_2 = \frac{B_2}{A_1}, \quad \gamma = \frac{Q_0}{U_0}, \quad \beta = \frac{k_0 L}{U_0}, \quad S = \frac{V_0}{\sqrt{U_0^2}}
\]

And upon the substitution of (6) in equations (2),(3) and (4), the governing equations transform into

\[
egin{align*}
 f'''' + 2f''f' - 2(f')^2 - Mf' &= 0 \\
 \left( 1 + \frac{4}{3} R \right) \theta'' + Pr(\phi' - f' - \epsilon_1 f' + \gamma \theta) &= 0 \\
 \phi'' + Sc(f\phi' - f'\phi - \epsilon_1 f' - \beta \phi) &= 0
\end{align*}
\]

and the boundary conditions take the following form:

\[
\begin{align*}
 f' &= 1, \quad f = S, \quad \theta = 1 - \epsilon_1, \quad \phi = 1 - \epsilon_2 \quad \text{at } \eta = 0 \\
 f' &\to 0, \theta \to 0, \phi \to 0 \quad \text{as } \eta \to \infty
\end{align*}
\]

where the prime denotes differentiation with respect to \( \eta \). \( M \) is the magnetic parameter, \( \epsilon_1 \) and \( \epsilon_2 \) are the thermal and solutal stratification parameters respectively, \( \epsilon_1, \epsilon_2 > 0 \) represents a stably stratified environment, while \( \epsilon_1 = \epsilon_2 = 0 \), corresponds to an unstratified environment, \( \gamma > 0 (or \ < 0) \) represents heat source(or sink) \( S > 0 (or \ < 0) \) is the suction(or blowing) parameter, \( \beta < 0 \) denotes generative chemical reaction, \( \beta > 0 \) denotes generative chemical reaction and \( \beta = 0 \) represents non-reactive species.

The above equations (7),(8) and (9) along with the boundary conditions (10) are solved by converting them to an initial value problem. We get

\[
\begin{align*}
 f' &= z, \quad z' = p, \quad p' = 2z^2 + Mz - fp \\
 \theta' &= q, \quad q' = -\left( \frac{3Pr}{3+4R} \right) (fq - z\theta - \epsilon_1 z + \gamma \theta)
\end{align*}
\]
\[ \phi' = r, \quad r' = -Sc(f r - z \phi - \epsilon_2 z - \beta \phi) \]  
(13)

With the boundary conditions

\[ f(0) = S, \quad f'(0) = 1, \quad \theta(0) = 1 - \epsilon_1, \quad \phi(0) = 1 - \epsilon_2 \]  
(14)

The resultant initial value problem is solved by employing Runge-Kutta fourth order technique. The step size \( \Delta \eta = 0.01 \) is used to obtain the numerical solution with decimal place accuracy as the criterion of convergence.

The parameters of practical interest for the present problem are the skin-friction coefficient, the Nusselt number and the Sherwood number, which are given respectively by the following expressions. Knowing the velocity field the skin-friction at the plate can be obtained, which in non-dimensional form is given by

\[ \frac{1}{2} \sqrt{Re} C_f = f'(0) \]

Knowing the temperature field, the rate of heat transfer coefficient can be obtained, which in non-dimensional form, in terms of Nusselt number is given by

\[ \sqrt{Re} Nu = -\theta'(0) \]

Knowing the concentration field, the rate of mass transfer coefficient can be obtained, which in non-dimensional form, in terms of Sherwood number, is given by

\[ \sqrt{Re} Sh = -\phi'(0) \]

where \( Re = \frac{u \rho L}{\mu} \) is the local Reynolds number.

### 4. Results and Discussion

To analyze the results, numerical computations has been carried out for variations in the governing parameters such as the magnetic field parameter \( M \), radiation parameter \( R \), thermal stratification parameter \( \epsilon_1 \), solutal stratification parameter \( \epsilon_2 \), suction / blowing parameter \( S \), heat source / sink parameter \( \gamma \), Prandtl number \( Pr \), Schimdt number \( Sc \), chemical reaction parameter \( \beta \). For illustration of these results, numerical values are plotted in figures 2-13.

In the present study following default parameter values are adopted for computations: \( M=0.1, \quad Pr=0.7, \quad \epsilon_1=0.2, \quad \gamma=0.1, R=0.1, \quad Sc=0.6, \quad \epsilon_2=0.2, \quad \beta=0.1, S=1 \). All graphs therefore correspond to these values unless specifically indicated on the appropriate graph.

![Figure 2: Effect of S on the velocity](image-url)
Figures (2),(3) and (4) display the effect of suction/blowing parameter S on velocity, temperature and concentration respectively. It is observed that velocity decreases significantly with increasing suction($S>0$) but increases with increasing blowing ($S<0$)(Figure 2). When the wall suction ($S>0$) is considered, this causes a decrease in the boundary layer thickness and velocity field is reduced. $S=0$ represents the case of a non-porous stretching sheet. It is known that imposition of wall suction ($S>0$) has the tendency to reduce the momentum boundary layer thickness, which causes reduction in the velocity. The opposite is noted for blowing ($S<0$). The physical explanation for such behavior is that
when stronger blowing is provided, the fluid is pushed farther from the wall where due to less influence of viscosity, the flow is accelerated. This effect acts to increase maximum velocity within the boundary layer. The same principle operates but in the reverse direction in the case of suction. Figure 3 and Figure 4 represent the temperature and concentration profiles for respectively for variable suction/blowing parameter S. It is seen that temperature as well as concentration decreases with increasing suction/blowing parameter. Fluid suction at the surface has a tendency to reduce the thermal as well as salutre boundary layer thickness on the other hand, the thermal and salutre boundary layer thickness increases with blowing. Thus, suction at the surface has a tendency to reduce hydro magnetic, thermal and salutre boundary layer thicknesses.

Figure 5: Effect of M on the velocity

Figure 6: Effect of M on the temperature
Figure 7: Effect of $M$ on the concentration

Figure 5 represents the velocity profiles for the variation of magnetic parameter $M$. With increasing values of $M$, fluid velocity is found to decrease. Actually, rate of transport decreases with the increase in $M$ because the Lorenz force which oppose the motion of fluid increases with the increase in $M$. From Figure 6 we see that the temperature increase with the increase of the magnetic field parameter, which implies that the applied magnetic field tends to heat the fluid and thus reduces the heat transfer from the wall. In Figure 7, the effect of an applied magnetic field is found to increase the concentration and hence increase the concentration boundary layer.

Figure 8: Effect of $\epsilon_1$ on the temperature
Figure 8 depict the effect of thermal stratification parameter $\epsilon_1$ on the temperature. It is found that the temperature decreases as the stratification parameter $\epsilon_1$ increases. This is quite obvious. Since increase in $\epsilon_1$ means increase in free stream temperature or decrease in surface temperature. Thermal boundary layer thickness is therefore also decreases with an increase in $\epsilon_1$ values.

Figure 9: Effect of $\gamma$ on the temperature

Figure 9 illustrate the influence of the heat generation (source) ($\gamma > 0$)/absorption (sink) ($\gamma < 0$) parameter $\gamma$ on the temperature. It is seen from the figure 8 that the temperature decreases with increasing heat absorption($\gamma < 0$) but increases with increasing heat generation ($\gamma > 0$), which implies owing to the presence of a heat source, the thermal state of the fluid increases causing the thermal boundary layer to increase.
The effect of the thermal radiation is presented in Figure 10. It is found that temperature increases as the radiation parameter $R$ increases. This is in agreement with the physical fact that the thermal boundary layer thickness increases with increasing $R$.

![Figure 10: Effect of Radiation on Temperature](image)

Figure 11: Effect of $Pr$ on temperature

Figure 11 depicts the effect of the Prandtl number $Pr$ on the temperature. It is observed that as the Prandtl number increases, the temperature decreases. If $Pr$ is small ($<1$), then the thermal diffusion occurs at a greater rate than the momentum diffusion and therefore the heat conduction is more effective than the convection. Conversely, if $Pr$ is large ($>1$), the momentum diffuses at a greater rate than the thermal diffusion and the convection is more effective than the conduction.

![Figure 12: Effect of $Sc$ on Concentration](image)

Figure 12: Effect of $Sc$ on the concentration
The influence of the Schmidt number $Sc$ on the concentration is plotted in Figure 12. The Schmidt number is the ratio of the momentum to the mass diffusivity. The Schmidt number therefore quantifies the relative effectiveness of momentum and mass transport by diffusion in the hydromagnetic (velocity) and concentration (species) boundary layers. As the Schmidt number increases the concentration decreases. This causes the concentration buoyancy effects to decrease yielding a reduction in the concentration boundary layer. This causes a reduction in concentration profiles.

![Figure 13: Effect of $\epsilon_2$ on the concentration](image)

Figure 13 is the graphical representation of concentration profiles for several values of salutal stratification parameter $\epsilon_2$. Concentration at a point is found to decrease as the stratification parameter increase $\epsilon_2$ increases. Concentration boundary layer thickness is there fore also decreased with an increase in $\epsilon_2$ values. It is observed that the concentration values are becoming negative inside the boundary layer for different values of the stratification parameter depending on the values of the other parameters. This is because the fluid near the plate can have concentration lower than the ambient medium.

For the verification of accuracy of the applied numerical scheme, a comparison of the present results corresponding to the values of heat transfer coefficient $[-\theta'(0)]$ for $R=0,\gamma=0$ (i.e., in absence of radiation and heat source/sink) with the available published results of Bidin and Nazar[7] in the absence of Eckert number (Ec) and thermal radiation(R) and Swati Mukhopadyaya [8] in the absence of thermal stratification $\epsilon_1$, suction S and Magnetic field paramer M is made and presented in Table 1. The results are found in good agreement.
Table 1: Comparison of the heat transfer coefficient of present case with those of Bidin and Nazar [7] and Swati Mukhopadyaya [8] for different values of \( Pr \).

<table>
<thead>
<tr>
<th>( Pr )</th>
<th>Bidin and Nazar[7] (( Ec=0=R ))</th>
<th>Swati Mukhopadyaya[8] (( \epsilon_1=0=S=M ))</th>
<th>Present study (( R=0=S=\gamma ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9547</td>
<td>0.9547</td>
<td>0.9747</td>
</tr>
<tr>
<td>2</td>
<td>1.4714</td>
<td>1.4714</td>
<td>1.4742</td>
</tr>
<tr>
<td>3</td>
<td>1.8691</td>
<td>1.8961</td>
<td>1.8647</td>
</tr>
</tbody>
</table>

Table 2: Effects suction/blowing parameter \( S \), thermal stratification parameter \( \epsilon_1 \), heat source/sink parameter \( \gamma \), Radiation parameter \( R \), Prandtl number \( Pr \) on the heat transfer coefficient \( -\theta'(0) \).

<table>
<thead>
<tr>
<th>( S )</th>
<th>( \epsilon_1 )</th>
<th>( \gamma )</th>
<th>( R )</th>
<th>( Pr )</th>
<th>( -\theta'(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.7</td>
<td>0.4999</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.7</td>
<td>0.6165</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9277</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.1</td>
<td>0.1</td>
<td>0.7</td>
<td>0.8573</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
<td>0.7</td>
<td>0.8319</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>-0.2</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9301</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>-0.2</td>
<td>0.2</td>
<td>0.7</td>
<td>0.8516</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>-0.2</td>
<td>0.2</td>
<td>1.25</td>
<td>1.3529</td>
</tr>
</tbody>
</table>

Table 2 shows the effects of the suction/blowing parameter \( S \), thermal stratification parameter \( \epsilon_1 \), heat source/sink parameter \( \gamma \), Radiation parameter \( R \), Prandtl number \( Pr \) on the heat transfer coefficient \( -\theta'(0) \). From this table it is clear that heat transfer coefficient increases with Prandtl number. This is because when \( Pr \) increases, the thermal diffusivity decreases and thus the heat is diffused away from the heated surface more slowly and in consequence increase the temperature gradient at the surface. From the table it is also noticed that with the increase of thermal stratification parameter \( \epsilon_1 \) there is a decrease in the heat transfer coefficient. The thermal boundary layer thickness decreases with an increase in the thermal stratification parameter \( \epsilon_1 \), due to stratification, the temperature in the boundary layer decreases, which results in decrease in the temperature gradient.

5. Conclusions

The governing equations for steady MHD two dimensional laminar mixed convective boundary layer flow of a viscous incompressible, electrically conducting and radiating fluid past an exponentially stretching porous sheet embedded in a doubly stratified medium with radiation and heat source/ sink effects was formulated. The resulting partial differential equations were transformed into a set of highly nonlinear ordinary differential equations using similarity transformations. Numerical solutions of these equations are obtained by fourth order Runge-Kutta method along with shooting technique. A comprehensive set of graphical results for the velocity, temperature and concentration is presented and their dependence on some physical parameters is discussed. The observations are:

(i) An increase in magnetic parameter decreases the velocity but increases temperature as well as concentration.
(ii) Suction at the surface has a tendency to reduce velocity, temperature and concentration.

(iii) An increase in radiation leads to an increase in the temperature. However, there is a decrease in heat transfer rate with an increase in radiation.

(iv) The temperature decreases with increasing values of the thermal stratification parameter. An increase in salutal stratification parameter decreases the concentration.

(v) Temperature increases with the heat source and temperature gradient decrease with heat source.

References